Cross Entropy: A New Solver for Markov Random Field Modeling and Applications to Medical Image Segmentation

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November 23, 2005

Overview

Purpose

Use **Cross Entropy** as a *general,stochastic* way to find an "optimal" solution for Markov Random Fields problems in Image Segmentation.

Markov Random Fields

- "synonymous" with Markov networks
- encode spatial information
- probability of an event at a position is dependant on neighbours
- *powerful*: can do the equivalent of smoothing and edge detection simultaneously
- *difficult to solve*: global optimum is often hard to find ("analytically intractable")

Solving Tough Problems

- *graph cuts*: polynomial time, but only if you can create a graph of the problem
- *simulated annealing*: "stochastic local search" — high temperature (randomness) to low temperature, but only optimal if you "cool" very slowly
- *belief propagation*: deterministic approximate solution, via conversion to a graph and propagating belief from a leaf "upwards"

Cross Entropy

- stochastic
  - Random
  - Longer you run, better the result
  - "Efficient"
- based on "well-established"/sound mathematical theory of rare events
- *Reuven Rubinstein*, Technion—Israel Institute of Technology, 1997

Rare Events

- want to get probability of rare event
- could do normal sampling
- Crude Monte Carlo
- random samples $X_1, \ldots, X_N$ from distribution X and the rare event $S(X_i) \leq \gamma$

$$\frac{1}{N}\sum_{i=1}^{N} I_{S(X_i)\leq\gamma}$$

  - if event is rare, this could take many samples
  - time-consuming, inaccurate

Importance Sampling

- can use importance sampling
  - aka biased sampling
  - sample the region near the rare event
  - adjust weight of results accordingly
- X_i is drawn from a pool with pdf $g(x)$, f(x) is the original distribution (i.e. flat)

$$\frac{1}{N}\sum_{i=1}^{N} I_{S(X_i)\leq\gamma} f(X_i)/g(X_i)$$

- to get better accuracy, $g(x)$ has to sample near the rare event
- cross entropy provides an iterative way of finding this region in configuration space

The Math

Want to optimise:

$$f^* = \operatorname*{argmin}_{f \in \mathcal{F}} E(f)$$

Do this by estimating biased probability of a “rare event”:

$$\mathcal{P}_v(E(F) \leq e) = \sum_f I_{\{E(f) \leq e\}} p(f; v)$$

where F is a vector of configurations with pdf $p(\cdot; v)$

- connect the equations by looking for $\mathcal{P}_v(E(f) \leq e)$ when $e = E(f^*)$ and the pdf is uniform
- rare event — global minimum is often unique

$$\mathcal{P}_v(E(f) \leq E(f^*)) = 1/|\mathcal{F}|$$

Cross Entropy Algorithm Overview

$$\mathcal{P}_v(E(F) \leq e) = \sum_f I_{\{E(f) \leq e\}} p(f; v)$$

- pretend you want to estimate $\mathcal{P}_v(E(F) \leq E(f^*))$ under a flat distribution for $p(\cdot; v)$ — use importance sampling
- when e is close to $E(f^*)$, a good sampling $p(\cdot; v)$ favours configurations near f^*
- *but* the closer e gets to $E(f^*)$, the harder it is to get good samples $\Rightarrow$ the better your choice of v must be
- cross entropy technique:
  - iteratively decrease e while improving v
  - power is that (for a well chosen family of pdfs) you can *analytically* choose the best possible v at each step

Image Segmentation Problem to solve:

- Label n sites using m labels.
- sites $\Leftrightarrow$ pixels
- labels $\Leftrightarrow$ number of different segments

CE Algorithm for MRF Energy Minimisation

Initialisation

- Set level $t = 1$
- The initial parameter vector $v_0 = \{v_{0,1}, \ldots, v_{0,n}\}$
- Each $v_{t,i} = \{v^1_{t,i}, \ldots, v^m_{t,i}\}$ is a vector with m elements for site i.

CE Algorithm for MRF Energy Minimisation

Main Loop

- Generate a collection of samples $F_1, \ldots, F_N$ ($F = \{f_1, ..., f_n\}$ is one MRF configuration) from the density $p(\cdot; v)$ and compute the energy $E_i = E(F_i)$ for every $i \in \{1, ..., N\}$.
- Sort all the E_i in a non-increasing order to $\{E_1, ..., E_N\}$. Then pick $e_t = E_{\lceil(1-\rho)N\rceil}$.
- Use the samples $F_1 \ldots F_N$ to update v_{t+1} using

$$v^j_{t+1,i} = \frac{\sum_{k=1}^{N} I_{\{E_k(F_k) \le e_t\}} I_{\{F_{ki}=j\}}}{I E_k(F_k) \le e_t}$$

  for $i = 1, \ldots, n$ and $j = 1, \ldots, m$.
- If e_t remains unchanged for several iterations, terminate loop; else, set $t = t + 1$ and repeat.

CE Algorithm for MRF Energy Minimisation

Termination

- The final $E_N(F_N)$ of T-th iteration is the estimated minimal MRF energy.

Notes on the Algorithm

- $p(\cdot; v)$ can be any probability distribution — in this case m-point Bernoulli is chosen (i.e. a label j is chosen for site i with probability $v^j_{t,i}$ at level t)
- parameters ρ and N
- ρ is the rarity of the event — tunable parameter, usually 1% to 10%. When n is large, choose ρ to be larger.
- N is the sample size. They choose $N = cn$, where c is between 1 and 10.
- alternative stopping criteria exist, such as v_t converging to a binary vector

The MRF Function

- maximise posterior $P(X|Y)$ where $P(X|Y) \propto P(X)P(Y|X)$
- X is labelling MRF, Y is "observed" (labelled) data
- multi-level logistics (MLL) and Gaussian distribution
- MRF Prior Energy:

$$U(X) = \sum_{i \in S} \sum_{j \in N_i} I_{\{x_i \neq x_j\}}$$

  where N_i is the neighbourhood (4 neighbours) and S is the set of all sites.
- Likelihood energy:

$$U(Y|X) = \sum_{i \in S} \sum_{j=0}^{m-1} I_{\{x_i = j\}} \cdot \frac{(y_i - \mu_j)^2}{2\sigma^2}$$

  where μ_j is the mean intensity of region j

Synthetic Test I

- Binary Segmentation of synthetic “Blood Vessels”
- Varying levels of noise, ground truth is known

Synthetic Test I — Results

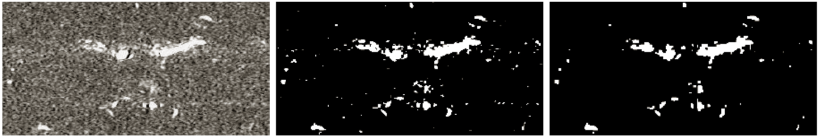
		BP		CE	
image	SNR	minimum	error	minimum	error
bar (width=3)	2	2215505	18.65%	1656942±7707	9.01%±1.53%
	3	1848604	7.54%	1577403±17069	3.52%±0.69%
	4	1598662	2.25%	1523299±17271	1.44%±0.96%
	5	1526652	0.59%	1501826±2126	0.22%±0.28%
bar (width=6)	2	2101877	19.29%	1319552±4612	1.20%±0.37%
	3	1659417	8.30%	1242550±1923	0.59%±0.20%
	4	1408863	3.13%	1232195±2309	0.03%±0.06%
	5	1250346	0.68%	1216728±2223	0.07%±0.12%
circle (width=3)	2	2241872	17.58%	1721610±6633	11.45%±1.04%
	3	2000580	8.59%	1691143±3865	6.72%±0.54%
	4	1817165	4.49%	1672465±2259	3.37%±0.40%
	5	1774078	2.34%	1675028±926	1.91%±0.20%
circle (width=6)	2	2178246	18.65%	1348983±3880	4.45%±1.11%
	3	1715610	8.30%	1295023±1162	1.98%±0.25%
	4	1516673	4.39%	1278273±3337	1.86%±0.08%
	5	1380865	1.86%	1269815±634	1.37%±0.08%

Synthetic Test II

- segmentation of BrainWeb T1 data
- 4 classes: white matter, grey matter, cerespinal fluid, other
- noise added

	BP		CE	
noise level	minimum	error	minimum	error
3%	348740	4.25%	211878.3 ± 2737.2	$2.77\% \pm 0.08\%$
5%	488391	6.00%	257021.0 ± 1505.7	$4.14\% \pm 0.06\ \%$
7%	487083	6.58%	229624.5 ± 1184.7	$3.57\% \pm 0.02\%$
9%	691409	11.65%	525039.5 ± 768.6	$9.08\% \pm 0.01\%$

- PCMRA: Phase contrast magnetic resonance angiographic images
- get lower mean energy
- no ground truth



Clinical Cerebral Data

- from IBSR: Internet Brain Segmentation Repository
- 3 labels: white matter, grey matter, other

	BP		CE	
dataset	minimum	error	minimum	error
1	19361	3.32%	15103 ± 39	$2.29\% \pm 0.08\%$
2	28049	4.85%	24665 ± 24	$3.70\% \pm 0.14\%$
3	34641	7.72%	30657 ± 28	$6.87\% \pm 0.24\%$
4	31483	4.98%	25458 ± 12	$2.94\% \pm 0.02\%$
5	22104	3.08%	18008 ± 15	$1.80\% \pm 0.08\%$
6	35652	8.27%	31274 ± 59	$5.50\% \pm 0.16\%$
7	37561	8.07%	31646 ± 39	$6.23\% \pm 0.24\%$
8	29782	4.72%	25317 ± 24	$2.86\% \pm 0.16\%$

Conclusions

- simple algorithm
- easy initialisation (can choose flat distribution)
- can solve any energy functional that can be evaluated
- able to find better optima
- inherently parallel